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93202Q

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

New Zealand Scholarship

Calculus, 2004

2.00 pm Wednesday 24 November 2004

QUESTION BOOKLET

You should answer ALL the questions in the separate Answer Booklet No. 93202A.

A 4-page booklet (S–CALCF) containing mathematical formulae and tables has been centre-stapled in the middle of this booklet. Before commencing, carefully detach the formulae and tables booklet.

Check that this Question Booklet has pages 2–7 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

Outcome Description

The student will use combinations of techiques and concepts from calculus, algebra, trigonometry, and geometry to solve complex problems in unfamiliar or generalised situations.

Scholarship Criteria

The student will:

- devise and/or use models to solve complex problems
- select appropriate combinations of techniques and concepts to solve complex problems.

Scholarship with Outstanding Performance Criteria

In addition to meeting the criteria for Scholarship, the student will:

- independently develop an extended chain of reasoning
- show insight and flair when solving complex problems
- demonstrate a high level of conceptual understanding through their ability to generalise and think abstractly.

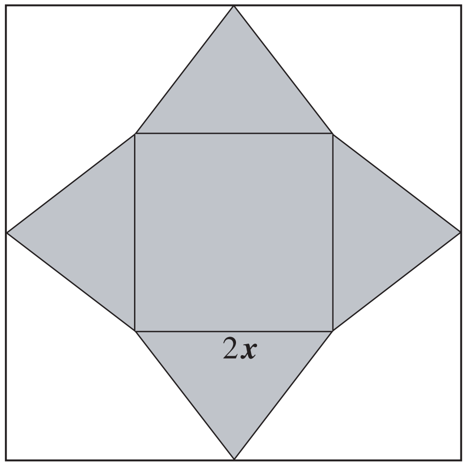
You should attempt ALL six questions. Show ALL your working.

QUESTION ONE

A square sheet of metal of side 2L centimetres has four pieces cut out symmetrically from the corners as shown in **Fig. 1**.

The remaining piece of metal is made up of a square of side $2x$ centimetres and four isosceles triangles, as shown.

Assume L is a constant and $\mathrm{L} > 2x$.



2L

Fig. 1

(a) Find the value of x that gives a maximum volume of the square-based pyramid that can be made from the shape in **Fig. 1**. You do not need to prove that it is a maximum.

(b) A pyramid, cut from a square sheet as described above, has a fixed base of side 2k (ie $x = \mathrm{k}$). This pyramid is turned upside down (see **Fig. 2**), so that the square is horizontal, and a hole is drilled in the top, as shown. Water is poured in through the hole so that the volume increases at a rate of $\dfrac{\mathrm{k}^2}{3\mathrm{L}}$ cm^3 per sec.

For what values of h is the height, h cm, of the water in the container increasing at a rate > $\dfrac{\mathrm{L}-2\mathrm{k}}{h+1}$ cm per sec?

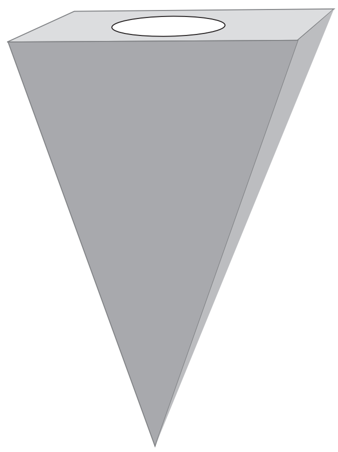


Fig. 2

QUESTION TWO

The parabola given by $y^2 = 4ax$ and the ellipse given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a>0$ and $b>0$, meet at the points P and Q, as shown in **Fig. 3**.

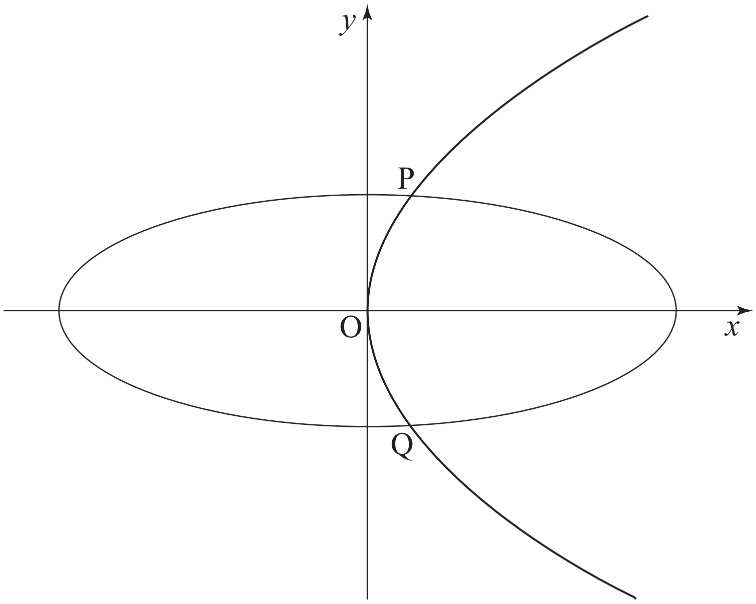


Fig. 3

(a) The two curves intersect in such a way that the tangent to the parabola at P is perpendicular to the tangent to the ellipse at P.

(i) Show that $b^2 = 2a^2$.

(ii) Hence, find in terms of a, the distance of the point P from the origin O.

(b) The tangent to the parabola at P meets the x-axis at M.
The tangent to the ellipse at P meets the x-axis at N.
Show that the length of $\mathrm{MN} = 2\sqrt{2}a$.

QUESTION THREE

(a) (i) **Fig. 4** below shows the function

$$y = \frac{x^2}{1+x^2} \qquad -1 \le x \le 1.$$

The gradient at the point $x = 1$ is $\frac{1}{2}$.

Hence show that there is a point with $\frac{1}{4} < x < \frac{1}{2}$, where the gradient is also $\frac{1}{2}$.

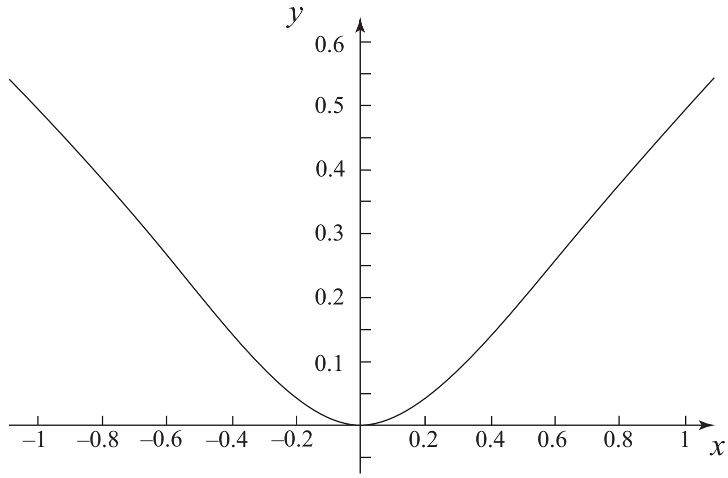


Fig. 4

(ii) The shape of a wooden ornament is made by rotating the area between the graph of the function

$$y = \frac{(x-1)^2}{1+(x-1)^2} \qquad 0 \le x \le 2$$

and the line $y = \frac{1}{2}$ through an angle 2π about the line $x = 1$.

Find the volume of this wooden ornament.

(b) (i) Show that $\int_0^a f(x)\mathrm{d}x = \int_0^a f(a-x)\mathrm{d}x$.

(ii) Hence, or otherwise, calculate the value of the following integral, showing clearly the steps in your working:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\mathrm{d}x, \text{ for any integer, } n.$$

QUESTION FOUR

(a) If $v = 1 + \mathrm{i}$ and $z = x + \mathrm{i}y$, for any real numbers x and y:

(i) Show that the equation

$$|z - v| = |vz|$$

represents a circle, and find its centre and radius.

(ii) Find the intersection of the circle in part (i) with the straight line

$$|z - v| = |z + v|.$$

(b) Using the roots of $z^5 = 1$, or otherwise, write $z^4 + z^3 + z^2 + z + 1$ as the product of two quadratic expressions with real coefficients.

Hence find the exact value of the product $\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right)$.

QUESTION FIVE

The nautilus is a marine creature that lives around coral reefs (**Fig. 5**).

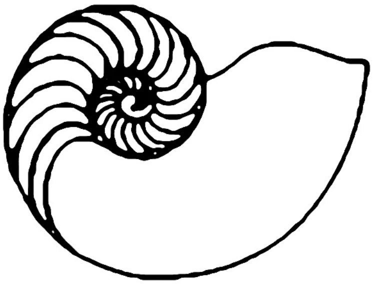


Fig. 5

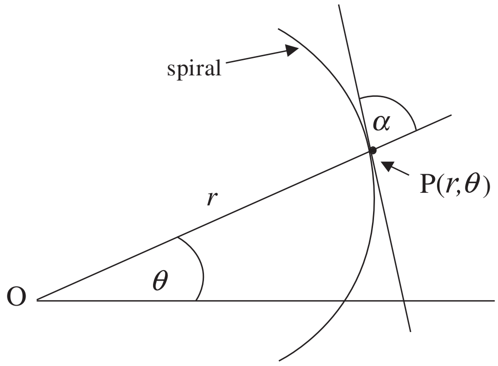


Fig. 6

The mathematical model of a nautilus shell (**Fig. 6**) is an equiangular spiral. Equiangular spirals have equations of the form $r = Ae^{k\theta}$, where k is a constant.

At every point P, the tangent to the curve makes the same angle, α, with the line OP from the point P to the origin (or pole), O.

The size of the angle α depends upon the number k in this mathematical model where $r = Ae^{k\theta}$.

(a) Using the parametric equations for the Cartesian co-ordinates (x,y) of the point P in terms of θ, find $\frac{dy}{dx}$.

(b) Hence, or otherwise, find the value of α in terms of k for this model.

QUESTION SIX

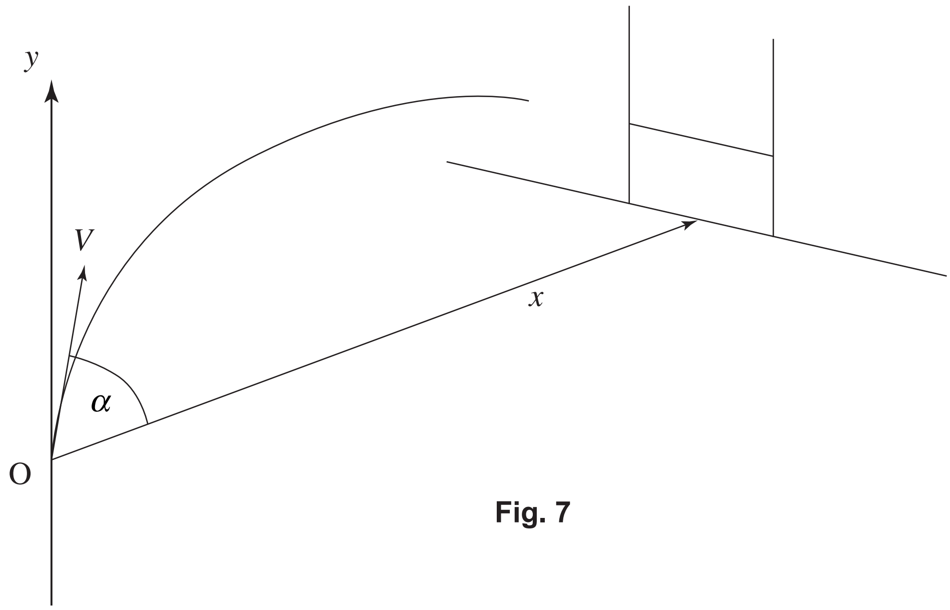


Fig. 7

(a) During a penalty conversion attempt a rugby ball may be considered to be a particle that moves as a projectile. Hence you may ignore the effects of air resistance, etc.
Initially, before the ball is kicked, the particle is considered to be at the origin, O, relative to a horizontal x-axis and a perpendicular y-axis (**Fig. 7**). Assume that the particle moves only in the plane of the x-y axes.

The acceleration of the particle after the ball has been kicked, measured relative to these two axes, is given by:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 0 \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -\mathrm{g}, \text{where g is the acceleration due to gravity.}$$

The particle leaves the ground with a speed of V m per sec at an angle of α to the x-axis.
By using integration, or otherwise, find the equation of the path of the particle.

(b) On a particular conversion attempt the co-ordinates of the centre of the goalpost crossbar are (kh,h).

(i) Show that there are two possible paths by which the particle may hit the centre of the crossbar

if $V^2 > \mathrm{g}h\left(1+\sqrt{1+k^2}\right)$.

(ii) In this case, show that for these two possible angles α_1 and α_2,

$$\alpha_1 + \alpha_2 = \tan^{-1}(-k).$$

ACKNOWLEDGEMENT
Question One (a) on page 2 is based on a question from *More Calculus*, page 10, © Nuffield Foundation, Ed Neil, H, published by Longman Group UK, 1994.